ATMAS Mathematics Specialist

2019 Test 2

Calculator Free

SHENTON COLLEGE

Name: SOLUTIONS

Time Allowed : 50 minutes

Marks	/59

Materials allowed: No special materials.

All necessary working and reasoning must be show for full marks.
Where appropriate, answers should be given in exact values.
Marks may not be awarded for untidy or poorly arranged work.

1 For a line passing through the point $\begin{pmatrix}5\\-1\end{pmatrix}$ and parallel to the vector $\begin{pmatrix}1\\4\end{pmatrix}$, find

a) The vector equation of the line. (1)

$$\tilde{r} = \begin{pmatrix}5\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\4\end{pmatrix}$$

b) The parametric equations of the line. (2)

$$\begin{cases} x = 5 + \lambda \\ y = -1 + 4\lambda \end{cases}$$

c) The Cartesian equation of the line. (2)

$\lambda = x - 5$ ✓ rearrange

$y = -1 + 4(x - 5)$ ✓ remove parameter

$y = 4x - 21$

2 Line L_1 has the vector equation $\mathbf{r} = \begin{pmatrix}2\\4\end{pmatrix} + \lambda\begin{pmatrix}-2\\3\end{pmatrix}$. Find the equation of L_2, a line perpendicular to L_1 and passing through position $\begin{pmatrix}1\\-5\end{pmatrix}$. (2)

$\begin{pmatrix}3\\2\end{pmatrix}$ is perpendicular to $\begin{pmatrix}-2\\3\end{pmatrix}$ ✓ perpendicular vector

$\therefore L_2 = \begin{pmatrix}1\\-5\end{pmatrix} + \mu\begin{pmatrix}3\\2\end{pmatrix}$ ✓ equation

3 If $f(x) = 9 - x^2$ and $g(x) = \sqrt{x+7}$, determine the domain and range of the composition $g(f(x))$. (4)

Domain $f(x)$ $x \in \mathbb{R}$. $-4 \le x \le 4$

Range $f(x)$ $y \le 9$ $y \ge -7$

✓ link range f to dom g

Domain $g(x)$ $x \le 9$ Natural domain $g(x)$ $x \ge -7$

Range $g(x)$ $y \le 4$ " range $y \ge 0$

✓ link nat dom g to range f.

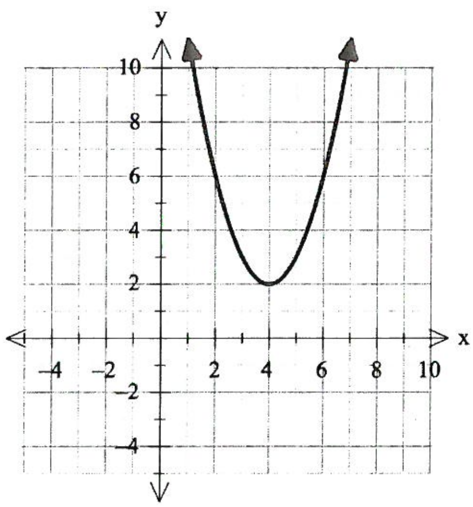
✓ Domain

✓ Range

$\therefore$ Domain $g(f(x))$ $\{x \in \mathbb{R}, -4 \le x \le 4\}$

Range $g(f(x))$ $\{y \in \mathbb{R}, 0 \le y \le 4\}$

4 The graph below shows the function $f(x) = (x-4)^2 + 2$.



a) Determine an appropriate restriction on the domain of $f(x)$ so that the inverse $f^{-1}(x)$ exists and is a decreasing function. (1)

$x \le 4$

b) Give the equation of the inverse based on your answer to part (a) in the form $y = \ldots$ (2)

$y = (x-4)^2 + 2, \quad x \le 4.$

$x = (y-4)^2 + 2, \quad y \le 4$ (domain of original becomes range of inverse).

✓ inverse process

$x - 2 = (y-4)^2$

✓ correct choice for ±

$\pm\sqrt{x-2} = y - 4$

$4 \pm \sqrt{x-2} = y$

Choose $y = 4 - \sqrt{x-2}$ to match $y \le 4$.

5 The function f(x) is defined as $f(x) = |x+3| + |x-1|$

a) Remove the absolute value signs by writing the function in piecewise form. (3)

$$f(x) = \begin{cases} -2x-2 & x < -3 \\ 4 & -3 \le x \le 1 \\ 2x+2 & x > 1 \end{cases}$$

✓ each function with domain

$x < -3$

$-(x+3) - (x-1)$

$= -2x - 2$

$-3 \le x \le 1$

$(x+3) - (x-1)$

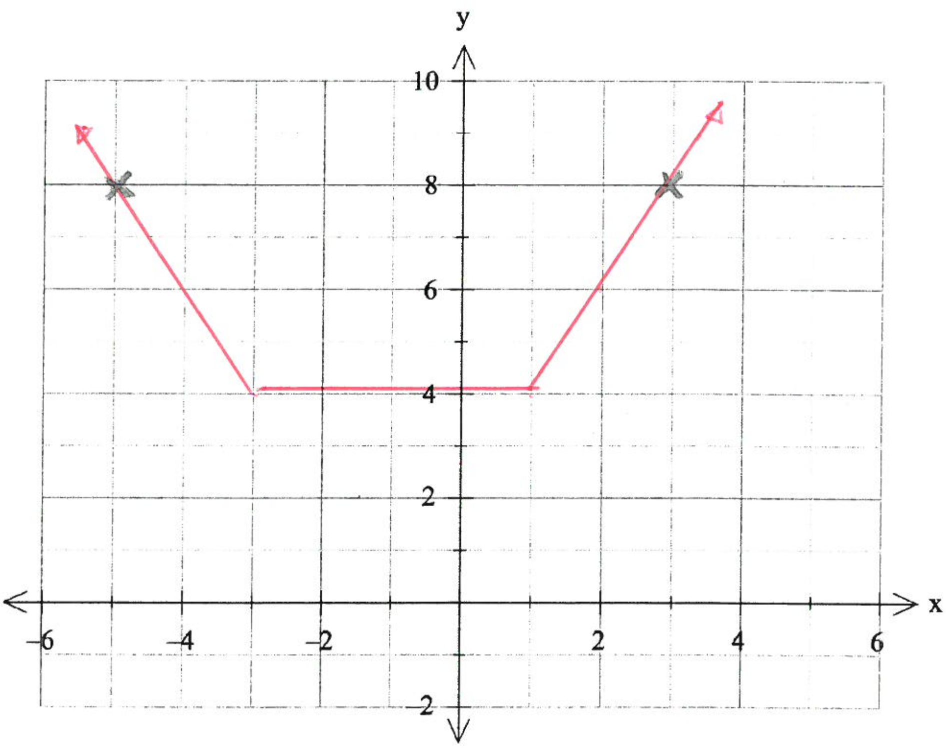
$= 4$

$x > 1$

$(x+3) + (x-1)$

$= 2x + 2$

b) Sketch the function $f(x) = |x+3| + |x-1|$ on the set of axes below. (3)



✓ each section matches (a)

c) Hence of otherwise solve $|x+3| + |x-1| = 8$ (1)

$x = -5$ or $x = 3$

6 The position vectors $\begin{pmatrix}5\\2\\2\end{pmatrix}$, $\begin{pmatrix}-3\\4\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\6\\-5\end{pmatrix}$ are all points on the plane P_1.

a) Determine the vector equation of P_1 using appropriate parameters. (3)

$$\begin{pmatrix}5\\2\\2\end{pmatrix}-\begin{pmatrix}-3\\4\\-1\end{pmatrix}=\begin{pmatrix}8\\-2\\3\end{pmatrix}$$

Vectors are not parallel

also $\begin{pmatrix}3\\-4\\7\end{pmatrix}$

$$\begin{pmatrix}-3\\4\\-1\end{pmatrix}-\begin{pmatrix}2\\6\\-5\end{pmatrix}=\begin{pmatrix}-5\\-2\\4\end{pmatrix}$$

$$P_1=\begin{pmatrix}5\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}8\\-2\\3\end{pmatrix}+\mu\begin{pmatrix}-5\\-2\\4\end{pmatrix}$$

✓ vector in plane

✓ second non-parallel vector in plane

✓ equation

b) Determine the Cartesian equation of P_1. (4)

i	j	k	i	j
8	-2	3	8	-2
-5	-2	4	-5	-2

$$\begin{pmatrix}8\\-2\\3\end{pmatrix}\times\begin{pmatrix}-5\\-2\\4\end{pmatrix}=\begin{pmatrix}-2\\-47\\-26\end{pmatrix}$$

✓✓ cross product.

$$\begin{pmatrix}2\\47\\26\end{pmatrix}\cdot\begin{pmatrix}5\\2\\2\end{pmatrix}=10+94+52$$
$$=156$$

✓ scalar product for constant.

Switched to positive just because

$$P_1:\ \tilde{r}\cdot\begin{pmatrix}2\\47\\26\end{pmatrix}=156$$
$$\Rightarrow\ 2x+47y+26z=156$$

✓ Cartesian.

7 If $h(x)=\frac{1}{8^x}$ and $h\big(k(x)\big)=2^{3-3x-3x^2}$, find the equation of $k(x)$. (4)

$$h(x)=2^{-3x}$$

$$\therefore\ h(k(x))=2^{-3k(x)}$$

$$-3k(x)=3-3x-3x^2$$

$$-3k(x)=-3(x^2+x-1)$$

$$k(x)=x^2+x-1$$

✓ h(x) in exponent form.

✓ substitute for k(x)

✓ equate powers

✓ k(x)

8 Consider the rational function $y = \frac{3x^2}{x-1}$. Given that $\frac{d^2y}{dx^2} = \frac{6}{(x-1)^3}$, draw a sketch of the function, indicating on your sketch important features such as asymptotes, intercepts, and critical points. (6)

Vertical asymptote at $x = 1$ ✓vertical

$$\frac{3x(x-1) + 3(x-1) + 3}{x-1} = 3x + 3 + \frac{3}{x-1}$$

oblique asymptote $y = 3x + 3$ ✓oblique

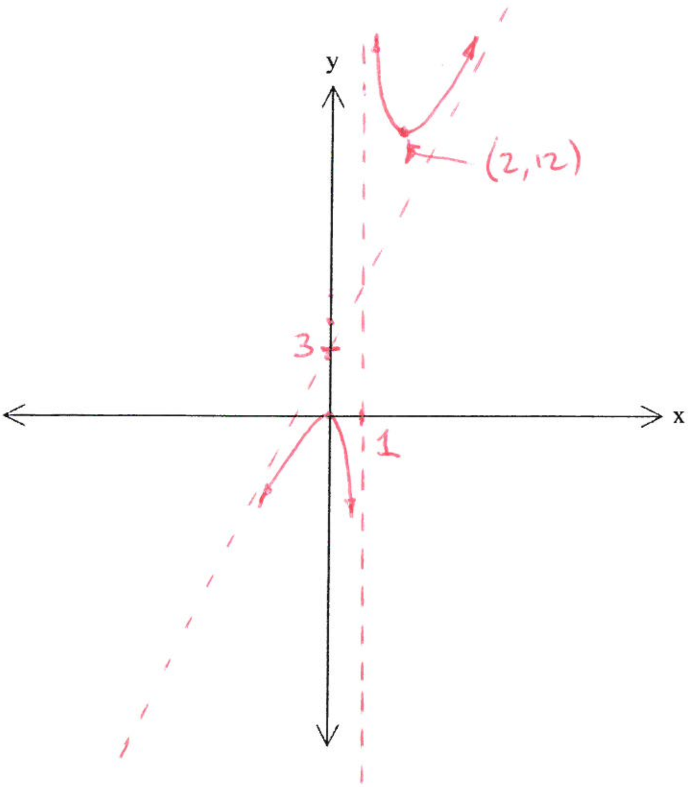
$$\frac{dy}{dx} = 3 - \frac{3}{(x-1)^2}$$

$\frac{dy}{dx} = 0$ when $x = 0$ or $x = 2$ ✓ stationary points

$y = 0$ $y = 12$

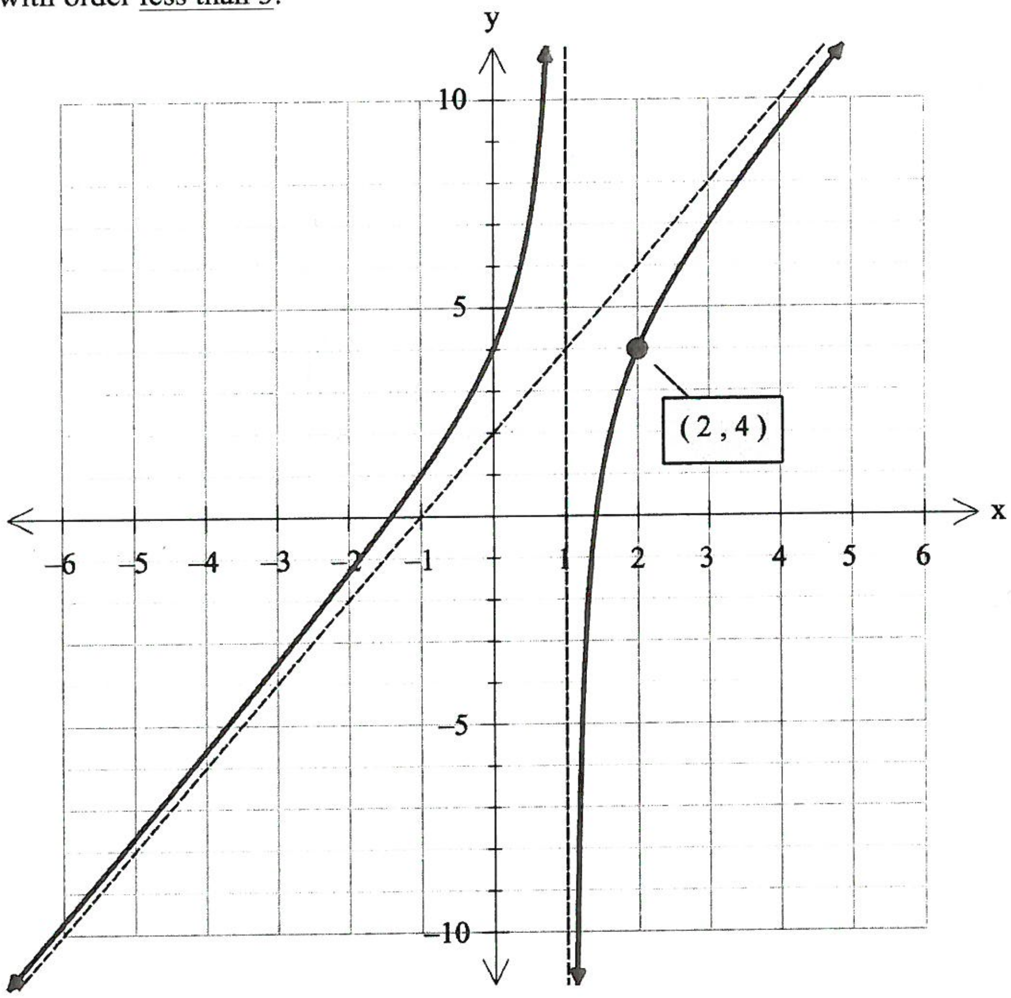
$\left.\frac{d^2y}{dx^2}\right|_{x=0} < 0$, $\Rightarrow$ max

$\left.\frac{d^2y}{dx^2}\right|_{x=2} > 0$, $\Rightarrow$ min. ✓ nature of stationary points



✓✓ graph.

9 The graph below was created from the function $y = \frac{f(x)}{g(x)}$. Both $f(x)$ and $g(x)$ are functions with order less than 3.



Determine both $f(x)$ and $g(x)$. (5)

from vertical asymptote, $g(x) = x - 1$

(can't be $(x-1)^2$ because then $f(x)$ would be order 3)

oblique asymptote $y = 2x+2$

So $y = 2x + 2 + \frac{r}{x-1}$

@ $(2,4)$ $\quad 4 = 2(2) + 2 + \frac{r}{2-1}$

$\Rightarrow r = -2$

$y = 2x + 2 - \frac{2}{x-1}$

$= \frac{(2x+2)(x-1) - 2}{x-1}$

$= \frac{2x^2 - 4}{x-1}$

$\therefore f(x) = 2x^2 - 4$

$g(x) = x - 1$

✓ vertical → $g(x)$

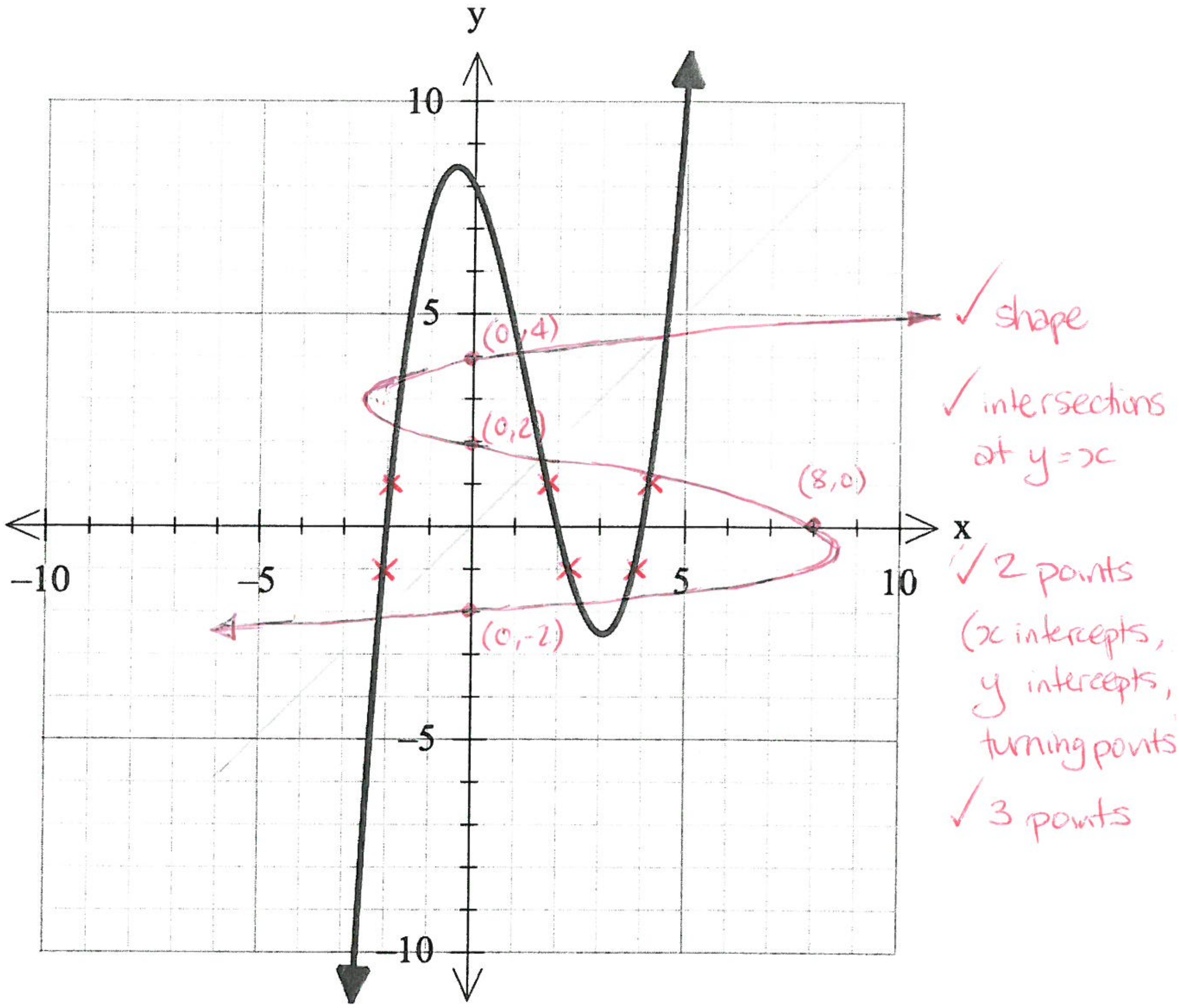
✓ equation for oblique.

✓ oblique + remainder form.

✓ solve for r

✓ $f(x)$

10 The graph below shows $y = f(x)$.



a) Mark on the graph the points where $f(x)$ would intersect with $\frac{1}{f(x)}$. (Do not graph $\frac{1}{f(x)}$.) (2)

✓ $y = 1$
✓ $y = -1$.

b) Add a sketch of $f^{-1}(x)$ to the axes above, indicating at least 3 key points. (4)

c) Explain why $f^{-1}(x)$ is not a function. (1)

$f^{-1}(x)$ is one-to-many, fails vertical line test, etc...

11

a) Determine any points of intersection between the sphere $\left| r - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right| = 3$ and the line (4)

$$r = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

✓ substitute line into sphere

$$\left| \begin{pmatrix} -2+\lambda \\ 3-\lambda \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right| = 3$$

✓ expand magnitude

$$(-7+\lambda)^2 + (4-\lambda)^2 = 9$$

✓ solve for λ

$$49 - 14\lambda + \lambda^2 + 16 - 8\lambda + \lambda^2 = 9$$

$$56 - 22\lambda + 2\lambda^2 = 0$$

$$(\lambda - 7)(\lambda - 4) = 0$$

✓ coordinates

$$\lambda = 7 \text{ or } \lambda = 4$$

$$r = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$$

b) Calculate the shortest distance between the sphere $\left| r - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right| = 3$ and the plane $r \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = -68$ (5)

Line from centre of sphere $\perp$ to plane

✓ line $\perp$ to plane through C

$$\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 5+3\lambda \\ -1 \\ 2-4\lambda \end{pmatrix}$$

✓ intersect line with plane

$$\begin{pmatrix} 5+3\lambda \\ -1 \\ 2-4\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = -68$$

$$15 + 9\lambda - 8 + 16\lambda = -68$$

✓ solve λ

$$\lambda = -3$$

$$\left| \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \right| = 5$$

✓ $|n|$ (or coord of intersection)

$\therefore$ distance from C to plane $= 3 \times 5 = 15$

radius 3 $\therefore$ distance from sphere to plane $= 12$

✓ distance.